FUNCTIONAL ANALYSIS

ICTP - 2020

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Exam 1 - February 20, 2020.

- Choose 5 out of the following 6 problems.
- Each problem is worth 5 points.
- Duration: 2h30min.

Problem 1. Let E be a Banach space. Let $T : E \to E$ and $S : E^* \to E^*$ be linear operators such that $f(Tx) = (Sf)(x)$ for any $x \in E$ and $f \in E^*$. Show that T and S are continuous.

Problem 2. Explain how to construct an element in $(\ell^\infty)^*$ that is not Jx for $x \in \ell^1$.

Problem 3.

(i) Let E be a Banach space and $K \subset E$ be a convex set. Prove that K is closed in the strong topology if and only if K is closed in the weak topology.

(ii) State Mazur's lemma (on the weak convergence and convex combinations).

(iii) Let $1 < p < \infty$. Let $\{f_n\}_{n\geq 1} \subset L^p(\mathbb{R})$ be a bounded sequence. Assume that $f_n \to g$ pointwise almost everywhere, for some $g \in L^p(\mathbb{R})$. Prove or disprove: $f_n \rightharpoonup g$ weakly.

Problem 4. Let $\ell^2 = \{a = (a_1, a_2, a_3, ...); a_j \in \mathbb{R}; \ \sum_{j=1}^{\infty} |a_j|^2 < \infty\}$ be the usual real vector space of square summable sequences, with norm given by

$$\|a\| = \left(\sum_{j=1}^{\infty} |a_j|^2\right)^{1/2}.$$

Prove or disprove: For every $0 \leq C \leq 1$ there is a sequence $\{x_n\}$ of elements in ℓ^2 such that x_n converges weakly to some $x \in \ell^2$ and

$$\|x\| = C \liminf_{n\to\infty} \|x_n\|.$$

Problem 5. Let X and Y be Banach spaces. Let $T : X \to Y$ be a bounded linear map and let $B_X = \{x \in X : \|x\|_X \leq 1\}$.

(i) Assume that $\overline{T(B_X)}$ is compact in Y (the closure here is with respect to the strong topology). Prove that if $x_n \rightharpoonup x$ weakly in X then $Tx_n \to Tx$ strongly in Y.

(ii) Assume that X is reflexive and that T satisfies the following property: if $x_n \rightharpoonup x$ weakly in X then $Tx_n \to Tx$ strongly in Y. Prove that $\overline{T(B_X)}$ is compact in Y.

Problem 6. Consider the Banach space $C[0,1]$ of the continuous functions $f : [0,1] \to \mathbb{R}$ with the supremum norm, i.e. $\|f\| = \sup_{x\in[0,1]} |f(x)|$. Let F be a closed subspace of $C[0,1]$ of infinite dimension. Prove that $F \not\subseteq C^1[0,1]$.

Note: $C^1[0,1]$ denotes the space of functions $f : [0,1] \to \mathbb{R}$ that are differentiable (with lateral derivatives at the extremals of the interval) such that $f' \in C[0,1]$.

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