FUNCTIONAL ANALYSIS

ICTP - 2019

INSTRUCTOR: EMANUEL CARNEIRO

Problem Set 1

Problem 1. Let $1 \leq p \leq \infty$. Prove that $L^p(\mathbb{R}^d)$ is a complete space.

Problem 2. Let $1 \leq p < r < q \leq \infty$. Assume that $f \in L^p(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$. Show that

$$\|f\|_r \leq \|f\|_p^{\theta} \, \|f\|_q^{1-\theta}\,,$$

where $0 \leq \theta \leq 1$ is such that

$$\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}.$$

Hence $f \in L^r(\mathbb{R}^d)$.

Problem 3. Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ be a measurable function that verifies the following property: for every d-dimensional rectangle Q one has

$$\left| \int_Q \varphi(x)\, \mathrm{d}x \right| \leq \frac{M\, m(Q)}{1 + m(Q)},$$

for a certain constant M, independent of Q. Show that for each $f \in L^1(\mathbb{R}^d)$ we have

$$\lim_{k\to\infty} \int_{\mathbb{R}^d} \varphi(kx)\, f(x)\, \mathrm{d}x = 0.$$

Problem 4. Prove Young's inequality for convolutions: let $1 \leq p, q, r \leq \infty$ be such that

$$1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}.$$

If $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$, then $f * g \in L^r(\mathbb{R}^d)$ and

$$\|f * g\|_r \leq \|f\|_p \, \|g\|_q.$$

Date: 31 de janeiro de 2019.
2000 *Mathematics Subject Classification.* XX-XXX.
Key words and phrases. XXX-XXX.

Problem 5. Let $f(x) = |x|^{-2}(1+|x|^2)^{-1}$ for $x \in \mathbb{R}^3$, equipped with the usual Lebesgue measure.

(i) For which values of $p \in (0, \infty]$ do we have $f \in L^p(\mathbb{R}^3)$?

(ii) Let $g \in L^2(\mathbb{R}^3)$. Prove that $f * g \in L^2(\mathbb{R}^3)$ and determine an explicit constant C such that C

$$\|f * g\|_{L^2(\mathbb{R}^3)} \leq C\|g\|_{L^2(\mathbb{R}^3)}.$$

Problem 6. Let $f \in L^2(\mathbb{R}^2)$ and define

$$g_n(x) = (\phi_n * f)(x) = \int_{\mathbb{R}^2} \phi_n(x-y)\, f(y)\, \mathrm{d}y, \qquad \phi_n(x) = n^4\, x_1\, x_2\, e^{-n|x|},$$

for all $x = (x_1, x_2) \in \mathbb{R}^2$, where $|x| = \sqrt{x_1^2 + x_2^2}$.

(i) Show that $\{g_n\}$ is a Cauchy sequence in $L^2(\mathbb{R}^2)$ and determine its limit.

(ii) What about if $\phi_n(x) = n^4\, |x_1|\, |x_2|\, e^{-n|x|}$. What would be the limit?

Problem 7. For $y > 0$ define

$$\varphi_y(x) = \frac{1}{\pi} \sin\left(\frac{\pi x y}{x^2+y^2}\right) \frac{y}{x^2+y^2};$$

$$\varphi_y * f(x) = \int_{-\infty}^{\infty} \varphi_y(x-z)\, f(z)\, \mathrm{d}z, \quad f \in L^2(\mathbb{R}).$$

(i) Show that $K(x,y) = \sqrt{y}\,(\varphi_y * f)(x)$ is a bounded function of x and y with $(x,y) \in \mathbb{R} \times (0,\infty)$.

(ii) For each $y > 0$, show that $\varphi_y * f \in L^2(\mathbb{R})$. Show also that $\varphi_y * f$ converges strongly (in $L^2(\mathbb{R})$) when $y \to 0$, and find this limit.

Problem 8. Let A and B be measurable subsets of $\mathbb{R}$, each with positive measure. Show that the sum-set

$$A + B = \{x + y;\, x \in A,\, y \in B\}$$

contains a segment.

Problem 9. Let $I = [-M, M] \subset \mathbb{R}$ and let h be a continuous function supported in I. Define

$$g(z) = \int_I \frac{1}{t-z}\, h(t)\, \mathrm{d}t$$

(i) Prove that g is analytic in $\mathbb{C}/I$.

(ii) Compute the limit $\lim_{|z|\to\infty} z g(z)$.

(iii) For $\varepsilon > 0$ and $\sigma \in \mathbb{R}$, compute the limit

$$\lim_{\varepsilon \to 0} \{g(\sigma + i\varepsilon) - g(\sigma - i\varepsilon)\}.$$

ICTP - Strada Costiera 11, Trieste, Italy, 34151.
Email address: carneiro@ictp.it