FUNCTIONAL ANALYSIS

ICTP - 2019

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Problem Set 2

Problem 10. Let $\Phi \in L^1(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} \Phi(x)\,\mathrm{d}x = A$. Assume that $|\Phi|$ admits a radial decreasing majorant $\psi \in L^1(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} \psi(x)\,\mathrm{d}x = B$. If $f \in L^p(\mathbb{R}^d)$, with $1 \leq p \leq \infty$, show that

$$\sup_{\varepsilon>0} |f * \Phi_\varepsilon(x)| \leq B\,(Mf)(x),$$

where M denotes the centered Hardy-Littlewood maximal operator. Use this fact (adapting the original proof of the Lebesgue Differentiation Theorem) to show that

$$\lim_{\varepsilon\to 0} f * \Phi_\varepsilon(x) = A\,f(x),$$

for almost every $x \in \mathbb{R}^d$.

(Hint: After trying a bit, if you cannot solve it, you may check and learn this fact in the book of Stein and Weiss' book "Fourier analysis in Euclidean spaces"or in Stein's book "Singular integrals and differentiability properties of functions". I believe this is a proposition there – with the complete proof for you).

Problem 11. The Gauss-Weierstrass kernel $K : \mathbb{R}^d \to \mathbb{R}$ is defined by

$$K(x) = \frac{1}{(4\pi)^{d/2}} e^{-|x|^2/4}.$$

For $t > 0$ define $K_t(x) = t^{-d/2} K(x/\sqrt{t})$. Let $f \in L^p(\mathbb{R}^d)$, with $1 \leq p \leq \infty$. Prove that $u(x,t) = f * K_t(x) \in C^\infty(\mathbb{R}^d \times (0,\infty))$ and that it verifies

$$u_t - \Delta u = 0 \ \text{ in } \ \mathbb{R}^d \times (0,\infty),$$

with

$$\lim_{t\to 0^+} u(x,t) = f(x) \ \text{ a.e. } x \in \mathbb{R}^d.$$

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Problem 12. The Poisson kernel $P : \mathbb{R}^d \to \mathbb{R}$ is defined by

$$P(x) = \frac{C_n}{(1+|x|^2)^{(d+1)/2}},$$

where $C_d = \Gamma\left(\frac{d+1}{2}\right)/\pi^{(d+1)/2}$. For $y > 0$ define $P_y(x) = y^{-d}P(x/y)$. Let $f \in L^p(\mathbb{R}^d)$, with $1 \leq p \leq \infty$. Prove that $u(x,y) = f * K_y(x)$ is harmonic in the upper half-space $\mathbb{R}^{d+1}_+ = \{(x,y);\ x \in \mathbb{R}^d;\ y > 0\}$ and that

$$\lim_{y \to 0^+} u(x,y) = f(x) \ \text{ a.e. } x \in \mathbb{R}^d.$$

Problem 13. For $g \in L^1_{loc}(\mathbb{R}^d)$ we define the Hardy-Littlewood centered maximal operator as

$$Mg(x) = \sup_{r>0} \frac{1}{m(B_r(x))} \int_{B_r(x)} |g(y)|\, \mathrm{d}y,$$

where $B_r(x) = \{y;\ |y-x| \leq r\}$ and $m(B_r(x))$ is its d-dimensional Lebesgue measure. Suppose that $d \geq 3$ and consider $f_\alpha(x) = |x|^{-\alpha}$, where $0 < \alpha < d$. Show that $Mf_\alpha(x) = C_\alpha f(x)$, $\forall x \in \mathbb{R}^d$, where C_α is a constant (that may depend on α).

Extra credit: Can you decide, for each $0 < \alpha < d$, if $C_\alpha = 1$ ou $C_\alpha > 1$? Justify. (For this you may need to learn a bit on harmonic functions from Evans' PDE book Chapter 2).

Problem 14. Let E be a Banach space. Prove that the unit ball $B = \{x \in E;\ \|x\| \leq 1\}$ is compact if and only if E has finite dimension.

Problem 15. Prove Hahn-Banach's Theorem in its complex version: let E be a vector space over $\mathbb{C}$ and $p : E \to \mathbb{R}$ a function such that:

(i) $p(ax) = |a|\, p(x)$ for all $a \in \mathbb{C}$ and $x \in E$.
(ii) $p(x+y) \leq p(x) + p(y)$, for any $x, y \in E$.

Let $G \subset E$ be a subspace and g a linear functional in G that verifies

$$|g(x)| \leq p(x)$$

for all $x \in G$. Then there exists a linear functional f in E that extends g and verifies

$$|f(x)| \leq p(x)$$

for all $x \in E$.

Problem 16. Use Zorn's lemma to prove the following:

(a) Every vector space E has a basis.
Note: Here a basis means an 'algebraic basis' (or Hamel basis), i.e. a set $\{e_i\}_{i \in I}$ such that each $x \in E$ can be written in a unique way as

$$x = \sum_{J \subset I} \lambda_i\, e_i, \quad \text{with } J \subset I, \quad J \ \text{ finite.}$$

(b) Every subspace $X \subset E$ admits a complementary subspace $Y \subset E$ such that

$$E = X \oplus Y\,,$$

i.e. each $e \in E$ can be written in a unique way as $e = x + y$, with $x \in X$ and $y \in Y$.

Problem 17. Brezis' book - Exercise 1.3

Problem 18. Brezis' book - Exercise 1.4.

Problem 19. Let $\chi : [-1,1] \to \mathbb{R}$ be the characteristic function of the interval $[0,1]$, i.e.

$$\chi(x) = \begin{cases} 1, & \text{se } \ 0 \leq x \leq 1 \\ 0, & \text{se } \ -1 \leq x < 0. \end{cases}$$

Prove that there exists a continuous linear functional

$$\varphi : L^\infty([-1,1]) \to \mathbb{R}$$

with norm 2 such that $\varphi(\chi) = 1$ and $\varphi(f) = 0$ for any continuous $f : [-1,1] \to \mathbb{R}$.

Problem 20. Let E be a Banach space (over $\mathbb{R}$, say). Let $u, v \in E$ be such that

$$\|u\| = \|v\| = 1 \quad \text{and} \quad \|2u + v\| = \|u - 2v\| = 3.$$

Show that there exists a linear functional $\varphi \in E^*$ of norm 1 such that

$$\varphi(u) = \varphi(v) = 1.$$

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