FUNCTIONAL ANALYSIS

ICTP - 2019

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Problem Set 3

Problem 21. Let E and F be normed vector spaces and $\mathcal{L}(E,F)$ be the space of continuous linear operators $T : E \to F$ with norm

$$\|T\|_{\mathcal{L}(E,F)} = \sup_{\|x\|_E \leq 1} \|Tx\|_F.$$

Show that if F is a Banach space then $\mathcal{L}(E,F)$ is a Banach space.

Problem 22.

(i) Prove the **Stone-Weierstrass theorem in the real case**: let X be a compact Hausdorff topological space and $\mathcal{A}$ be a closed algebra of functions in $C(X,\mathbb{R})$ (i.e. if $\mathcal{A}$ is a vector subspace and if $f, g \in \mathcal{A}$ then $fg \in \mathcal{A}$) that separates points (i.e. for every $x, y \in X$ there exists $f \in \mathcal{A}$ such that $f(x) \neq f(y)$). Then either $\mathcal{A} = C(X,\mathbb{R})$ or $\mathcal{A} = \{f \in C(X,\mathbb{C}); f(x_0) = 0\}$ for some $x_0 \in X$.

Note: the second option does not happen if $\mathcal{A}$ contains the constant functions.

(ii) Prove the **Stone-Weierstrass theorem in the complex case**: let X be a compact Hausdorff topological space and $\mathcal{A}$ be a closed algebra of functions in $C(X,\mathbb{C})$ such that: (a) $\mathcal{A}$ separates points; (b) $\mathcal{A}$ is closed under conjugation (i.e. if $f \in A$ then $\overline{f} \in A$). Then $\mathcal{A} = C(X,\mathbb{C})$ or $\mathcal{A} = \{f \in C(X,\mathbb{C}); f(x_0) = 0\}$ for some $x_0 \in K$.

Note: the second option does not happen if $\mathcal{A}$ contains the constant functions.

(iii) Prove that if we remove the condition (b) in (ii) above the result is not true. For instance, consider X to be the unit circle in the complex plane and prove that the continuous function $z \mapsto \overline{z}$ cannot be uniformly approximated by polynomials $P(z)$ in X.

Problem 23. Let (X,d) be a compact metric space. We say that a set $K \subset C(X)$ is equicontinuous if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$d(x,y) < \delta \Rightarrow |f(x) - f(y)| < \varepsilon\,,$$

for all $x, y \in X$ and $f \in K$. Prove the **Arzela-Ascoli theorem**: a subset $K \subset C(X)$ is compact if and only if it is closed, bounded and equicontinuous.

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Problem 24. Let $a = (a_n)_{n\geq 1} \in \ell^2$ a sequence such that $a_n \neq 0$ for all $n \geq 1$. Prove that there exists a sequence $b = (b_n)_{n\geq 1} \in \ell^1$ such that

$$\left(\frac{b_n}{a_n}\right)_{n\geq 1} \notin \ell^2.$$

Problem 25. Consider the Banach space $C(T) = \{f : [-\frac{1}{2}, \frac{1}{2}] \to \mathbb{C};\ f \text{ continuous}\}$ with the supremum norm. For $f \in C(T)$ define

$$\widehat{f}(k) = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi i k x}\, f(x)\, \mathrm{d}x$$

where $k \in \mathbb{Z}$, and consider

$$S_n f(x) = \sum_{k=-n}^{n} \widehat{f}(k)\, e^{2\pi i k x}.$$

Take $x = 0$ and define

$$T_n(f) = S_n f(0).$$

(a) Prove that each T_n is a bounded linear functional in $C(T)$.
(b) Prove that $\|T_n\| \to \infty$ when $n \to \infty$.
(c) Prove that there exists $f \in C(T)$ such that $S_n f(0)$ diverges when $n \to \infty$.

Problem 26. Consider the Banach space $C[0,1]$ of the continuous functions $f : [0,1] \to \mathbb{R}$ with the supremum norm, i.e. $\|f\| = \sup_{x\in[0,1]} |f(x)|$. Let F be a closed subspace of $C[0,1]$ of infinite dimension. Prove that $F \nsubseteq C^1[0,1]$.

Note: $C^1[0,1]$ denotes the space of functions $f : [0,1] \to \mathbb{R}$ that are differentiable (with lateral derivatives at the extremals of the interval) such that $f' \in C[0,1]$.

Problem 27. Prove or disprove:

(i) Let $k \in \mathbb{N}$. Then span $\{1,\, x^{kn}\,;\ \ n = 1, 2, ...\}$ is dense in $C([0,1], \mathbb{R})$.
(ii) Let p_n be the n-th prime number. Then span $\{1,\, x^{p_n}\,;\ \ n = 1, 2, ...\}$ is dense in $C([0,1], \mathbb{R})$. (Hint: You may take a look in the literature for a result called Müntz approximation theorem).

Problem 28. Let $(X, \mathcal{M}, \mu)$ be a finite measure space, i.e. $\mu(X) < \infty$. Suppose that:

(i) E is a closed subspace of $L^p(X, \mu)$, for some $1 \leq p < \infty$.
(ii) $E \subset L^\infty(X, \mu)$.

Show that E is finite dimensional.

Hint: Use the closed graph theorem to show that $\|\cdot\|_2$ and $\|\cdot\|_\infty$ are equivalent norms in E.

Problem 29. Brezis' book - exercise 2.4

Problem 30. Brezis' book - exercise 2.8

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