FUNCTIONAL ANALYSIS

ICTP - 2020

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Problem Set 3

Problem 21. (Characterization of compactness via sequences) Let (X, d) be a metric space. Show that X is compact if and only if every sequence $\{x_n\}_{n\geq 1} \subset X$ admits a subsequence $\{x_{n_k}\}$ that converges to an element in X.

Problem 22. (Tychonoff's theorem) Let $\{X_\alpha\}_{\alpha\in I}$ be a family of compact topological spaces, indexed by some set I. Show that the product space

$$X = \prod_{\alpha\in I} X_\alpha$$

is compact in the product topology.

Problem 23. Show the if E is finite dimensional, the strong and weak topologies are the same.

Problem 24. Let E and F be Banach spaces and $T : E \to F$ be a linear operator. Prove that T is continuous in the strong topologies (for E and F) if and only if T is continuous in the weak topologies (for E and F).

Problem 25. Give an example of a situation where $f_n \overset{*}{\rightharpoonup} f$ in E^*, $x_n \rightharpoonup x$ in E, but $\langle f_n, x_n\rangle \nrightarrow \langle f, x\rangle$.

Problem 26. Let X be a vector space and $\varphi, \varphi_1, ..., \varphi_k$ be $k+1$ linear functionals on E with the following property: if $v \in E$ is such that $\varphi_i(v) = 0$ for $i = 1, 2, \ldots, k$ then $\varphi(v) = 0$. Prove that there exist constants $\lambda_1, \lambda_2, \ldots, \lambda_k$ such that

$$\varphi = \sum_{i=1}^{k} \lambda_i \varphi_i.$$

Hint: Consider the map $F : E \to \mathbb{R}^{k+1}$ given by $F(v) = (\varphi(v), \varphi_1(v), \ldots, \varphi_k(v))$ and use Hahn-Banach in $\mathbb{R}^{k+1}$.

Problem 27. Let E be a Banach space. Let $\varphi : E^* \to \mathbb{R}$ be a linear and continuous functional. Assume that φ is continuous in the weak-$*$ topology. Prove that there exists $x \in E$ such that $\varphi = J_x$. (Hint: Use the previous exercise).

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Problem 28. Let E be a Banach space and let H be a hyperplane in E^*, i.e. a set of the form $H = \{f \in E^* \ ; \ \varphi(f) = \alpha\}$ for some $\varphi \in E^{**}$. Show that H is closed in the weak-$*$ topology $\sigma(E^*, E)$ of E^* if and only if $\varphi = J_x$ for some $x \in E$. Conclude that if the canonical injection $J : E \to E^{**}$ is not injective then the weak topology $\sigma(E^*, E^{**})$ and the weak-$*$ topology $\sigma(E^*, E)$ are different.

Problem 29. Brezis' book - exercise 3.4.

Problem 30. Brezis' book - exercise 3.8.

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