FUNCTIONAL ANALYSIS

ICTP - 2020

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Problem Set 5

Problem 41. Let $1 \leq p < r < q \leq \infty$. Assume that $f \in L^p(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$. Show that

$$\|f\|_r \leq \|f\|_p^{\theta} \, \|f\|_q^{1-\theta},$$

where $0 \leq \theta \leq 1$ is such that

$$\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}.$$

Hence $f \in L^r(\mathbb{R}^d)$.

Problem 42. Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ be a measurable function that verifies the following property: for every d-dimensional rectangle Q one has

$$\left| \int_Q \varphi(x) \, \mathrm{d}x \right| \leq \frac{M\, m(Q)}{1 + m(Q)},$$

for a certain constant M, independent of Q. Show that for each $f \in L^1(\mathbb{R}^d)$ we have

$$\lim_{k \to \infty} \int_{\mathbb{R}^d} \varphi(kx) \, f(x) \, \mathrm{d}x = 0.$$

Problem 43. Prove Young's inequality for convolutions: let $1 \leq p, q, r \leq \infty$ be such that

$$1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}.$$

If $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$, then $f * g \in L^r(\mathbb{R}^d)$ and

$$\|f * g\|_r \leq \|f\|_p \, \|g\|_q.$$

Problem 44. Let $f(x) = |x|^{-2}(1 + |x|^2)^{-1}$ for $x \in \mathbb{R}^3$, equipped with the usual Lebesgue measure.

(i) For which values of $p \in (0, \infty]$ do we have $f \in L^p(\mathbb{R}^3)$?

(ii) Let $g \in L^2(\mathbb{R}^3)$. Prove that $f * g \in L^2(\mathbb{R}^3)$ and determine an explicit constant C such that C

$$\|f * g\|_{L^2(\mathbb{R}^3)} \leq C \|g\|_{L^2(\mathbb{R}^3)}.$$

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Problem 45. Let $f \in L^2(\mathbb{R}^2)$ and define

$$g_n(x) = (\phi_n * f)(x) = \int_{\mathbb{R}^2} \phi_n(x-y)\, f(y)\, \mathrm{d}y, \qquad \phi_n(x) = n^4\, x_1\, x_2\, e^{-n|x|},$$

for all $x = (x_1, x_2) \in \mathbb{R}^2$, where $|x| = \sqrt{x_1^2 + x_2^2}$.

(i) Show that $\{g_n\}$ is a Cauchy sequence in $L^2(\mathbb{R}^2)$ and determine its limit.

(ii) What about if $\phi_n(x) = n^4\, |x_1|\, |x_2|\, e^{-n|x|}$. What would be the limit?

Problem 46. For $y > 0$ define

$$\varphi_y(x) = \frac{1}{\pi} \sin\left(\frac{\pi x y}{x^2 + y^2}\right) \frac{y}{x^2 + y^2};$$

$$\varphi_y * f(x) = \int_{-\infty}^{\infty} \varphi_y(x - z)\, f(z)\, \mathrm{d}z, \quad f \in L^2(\mathbb{R}).$$

(i) Show that $K(x,y) = \sqrt{y}\,(\varphi_y * f)(x)$ is a bounded function of x and y with $(x,y) \in \mathbb{R} \times (0,\infty)$.

(ii) For each $y > 0$, show that $\varphi_y * f \in L^2(\mathbb{R})$. Show also that $\varphi_y * f$ converges strongly (in $L^2(\mathbb{R})$) when $y \to 0$, and find this limit.

Problem 47. Let A and B be measurable subsets of $\mathbb{R}$, each with positive measure. Show that the sum-set

$$A + B = \{x + y;\, x \in A,\, y \in B\}$$

contains a segment.

Problem 48. Let $I = [-M, M] \subset \mathbb{R}$ and let h be a continuous function supported in I. Define

$$g(z) = \int_I \frac{1}{t - z}\, h(t)\, \mathrm{d}t$$

(i) Prove that g is analytic in $\mathbb{C}/I$.

(ii) Compute the limit $\lim_{|z| \to \infty} z g(z)$.

(iii) For $\varepsilon > 0$ and $\sigma \in \mathbb{R}$, compute the limit

$$\lim_{\varepsilon \to 0} \{g(\sigma + i\varepsilon) - g(\sigma - i\varepsilon)\}.$$

Problem 49. Let $\Phi \in L^1(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} \Phi(x)\, \mathrm{d}x = A$. Assume that $|\Phi|$ admits a radial decreasing majorant $\Psi \in L^1(\mathbb{R}^d)$, with $\int_{\mathbb{R}^d} \Psi(x)\, \mathrm{d}x = B$. If $f \in L^p(\mathbb{R}^d)$, with $1 \leq p \leq \infty$, show that

$$\sup_{\varepsilon > 0} |f * \Phi_\varepsilon(x)| \leq B\, (Mf)(x),$$

where M denotes the centered Hardy-Littlewood maximal operator. Use this fact (adapting the original proof of the Lebesgue Differentiation Theorem) to show that

$$\lim_{\varepsilon \to 0} f * \Phi_\varepsilon(x) = A f(x)$$

for almost every $x \in \mathbb{R}^d$.

(Hint: After trying a bit, if you cannot solve it, you may check and learn this fact in the book of Stein and Weiss' book "Fourier analysis in Euclidean spaces"or in Stein's book "Singular integrals and differentiability properties of functions". I believe this is a proposition there – with the complete proof for you).

Problem 50.

(i) The Gauss-Weierstrass kernel $K : \mathbb{R}^d \to \mathbb{R}$ is defined by

$$K(x) = \frac{1}{(4\pi)^{d/2}} e^{-|x|^2/4}.$$

For $t > 0$ define $K_t(x) = t^{-d/2} K(x/\sqrt{t})$. Let $f \in L^p(\mathbb{R}^d)$, with $1 \leq p \leq \infty$. Prove that $u(x,t) = f * K_t(x) \in C^\infty(\mathbb{R}^d \times (0,\infty))$ and that it verifies the heat equation

$$u_t - \Delta u = 0 \;\text{ in }\; \mathbb{R}^d \times (0,\infty),$$

with

$$\lim_{t \to 0^+} u(x,t) = f(x) \;\text{ a.e. } x \in \mathbb{R}^d.$$

(ii) The Poisson kernel $P : \mathbb{R}^d \to \mathbb{R}$ is defined by

$$P(x) = \frac{C_d}{(1+|x|^2)^{(d+1)/2}},$$

where $C_d = \Gamma\big(\frac{d+1}{2}\big)/\pi^{(d+1)/2}$. For $y > 0$ define $P_y(x) = y^{-d} P(x/y)$. Let $f \in L^p(\mathbb{R}^d)$, with $1 \leq p \leq \infty$. Prove that $u(x,t) = f * P_y(x)$ is harmonic in the upper-space $\mathbb{R}^{d+1}_+ = \{(x,y);\; x \in \mathbb{R}^d,\; y > 0\}$ and that

$$\lim_{y \to 0^+} u(x,y) = f(x) \;\text{ a.e. } x \in \mathbb{R}^d.$$

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