FUNCTIONAL ANALYSIS

ICTP - 2020

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Problem Set 6

Problem 51. Compute the Fourier transform of the Gaussian $G_\delta(x) = e^{-\pi\delta|x|^2}$ on $\mathbb{R}^d$, where $\delta > 0$.

Problem 52.

(i) Show that for each $\beta > 0$ we have:

$$e^{-\beta} = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} e^{-\frac{\beta^2}{4u}}\, \mathrm{d}u.$$

(ii) Deduce the Fourier transform of $f_\lambda(x) = e^{-2\pi\lambda|x|}$ in $\mathbb{R}^d$, where $\lambda > 0$.

Problem 53. Let f be a function in $C_c^2(\mathbb{R}^d)$. Prove that:

$$\sum_{\substack{i=1\\ j=1}}^{d} \int_{\mathbb{R}^d} \left| \frac{\partial^2}{\partial x_i \partial x_j} f(x) \right|^2 \mathrm{d}x \leq C \sum_{i=1}^{d} \int_{\mathbb{R}^d} \left| \frac{\partial^2}{\partial x_i \partial x_i} f(x) \right|^2 \mathrm{d}x.$$

What is the best constant C in this inequality?

Problem 54. Let α, β be multi-indices and a, b be real numbers. Suppose that for any $f \in \mathbb{C}_c^\infty(\mathbb{R}^d)$ we have

$$\||\xi|^a \partial^\alpha \widehat{f}\|_{L^q(\mathbb{R}^d)} \leq C \||x|^b \partial^\beta f\|_{L^p(\mathbb{R}^d)}.$$

What is the relation between $a, b, \alpha, \beta, p, q, d$?

Note: Above we use $f(x)$ and $\widehat{f}(\xi)$.

Problem 55. The purpose of the following exercise is to show that $\mathcal{F}(L^1(\mathbb{R}^d)) \subsetneq C_0(\mathbb{R}^d)$.

(i) In dimension $d = 1$, let $f \in L^1(\mathbb{R})$ and suppose that $\widehat{f}$ is an odd function. Show that

$$\left| \int_1^a \frac{\widehat{f}(x)}{x}\, \mathrm{d}x \right| \leq C,$$

for any $a > 1$, with C independent of a.

(ii) Find an odd function $g \in C_0(\mathbb{R})$ such that

$$\left| \int_1^a \frac{g(x)}{x}\, \mathrm{d}x \right|$$

is not bounded as $a \to \infty$.

Problem 56.

(i) Let $0 < \alpha < d$ and define $C_\alpha = \Gamma(\alpha/2)/\pi^{\alpha/2}$. Prove that the Fourier transform of $C_\alpha|x|^{-\alpha}$ is $C_{d-\alpha}|x|^{-d+\alpha}$ in the tempered distribution sense, that is, for every φ in the Schwartz class $\mathcal{S}$ we have
$$\int_{\mathbb{R}^d} C_{d-\alpha}|x|^{-d+\alpha}\widehat{\varphi}(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} C_\alpha|x|^{-\alpha}\varphi(x)\,\mathrm{d}x.$$
Hint: As we did for the Fourier transform of the exponential, it is possible to relate these functions with the Gaussian. Show that
$$\int_0^\infty e^{-\pi\lambda|x|^2}\,\lambda^{\beta-1}\,\mathrm{d}\lambda = (\pi|x|^2)^{-\beta}\,\Gamma(\beta),$$
for $\beta > 0$.

(ii) Compute explicitly $g(x) = |x|^{-\alpha} * |x|^{-\beta}$, where $0 < \alpha, \beta < d$, and $d < \alpha + \beta < 2d$.

Problem 57. Let f be a smooth and integrable function on $\mathbb{R}$ such that f and $\widehat{f}$ decay faster than any polynomial (that is, $x^k f(x) \to 0$ as $|x| \to \infty$ for any k, and $\xi^k \widehat{f}(\xi) \to 0$ as $|\xi| \to \infty$ for any k). Prove that f belongs to the Schwartz class.

Problem 58. Let $\operatorname{sinc} x = (\sin \pi x)/\pi x$, with $\operatorname{sinc} 0 = 1$.

(i) If $a > 0$, show that $\widehat{\chi}_{[-a,a]}(\xi) = \check{\chi}_{[-a,a]}(\xi) = 2a \operatorname{sinc} 2a\xi$.

(ii) Let $\mathcal{H}_a = \{f \in L^2(\mathbb{R});\ \widehat{f}(\xi) = 0 \text{ (a.e) for } |\xi| > a\}$. Show that $\mathcal{H}_a$ is a Hilbert space and that $\{\sqrt{2a}\operatorname{sinc}(2ax - k); k \in \mathbb{Z}\}$ is an orthonormal basis of $\mathcal{H}_a$.

(iii) Show that if $f \in \mathcal{H}_a$, then $f \in C_0(\mathbb{R})$ (possibly after modification on a set of measure zero), and that
$$f(x) = \sum_{k=-\infty}^{\infty} f\left(\frac{k}{2a}\right) \operatorname{sinc}(2ax - k),$$
where this series converges uniformly (and also in L^2).

Problem 59. (Heisenberg's uncertainty). Let f be a Schwartz function on $\mathbb{R}$. Show that
$$\left(\int_{\mathbb{R}} |f(x)|^2\,\mathrm{d}x\right)^2 \le 16\pi^2 \int_{\mathbb{R}} |xf(x)|^2\,\mathrm{d}x \int_{\mathbb{R}} |\xi\widehat{f}(\xi)|^2\,\mathrm{d}\xi.$$
Hint: Use integration by parts to get $\int |f|^2 = -2\mathrm{Re}\int x f f'$.

Problem 60. In 1975, W. Beckner found the sharp form of the Hausdorff-Young inequality:
$$\|\widehat{f}\|_{p'} \le C(p)\|f\|_p,$$
where $1 \le p \le 2$ and $C(p) = \left[p^{1/p}/p'^{1/p'}\right]^{d/2}$. Differentiating the sharp form of the Hausdorff-Young inequality (with respect to the parameter p) at $p = 2$, show that if $f \in \mathcal{S}(\mathbb{R}^d)$ with $\|f\|_2 = 1$ we have
$$-\int_{\mathbb{R}^d} \ln|f(x)||f(x)|^2\,\mathrm{d}x - \int_{\mathbb{R}^d} \ln|\widehat{f}(\xi)||\widehat{f}(\xi)|^2\,\mathrm{d}\xi \ge \frac{n}{2}(1 - \ln 2).$$

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