REAL ANALYSIS

ICTP - 2020

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FINAL EXAM - DEC 03, 2020.

- Each problem is worth 5 points.
- Duration: 3h 30min

Problem 1. Let $f : \mathbb{R} \to \mathbb{C}$ be an integrable function. Define its Fourier transform $\widehat{f} : \mathbb{R} \to \mathbb{C}$ by

$$\widehat{f}(t) = \int_{\mathbb{R}} e^{-2\pi i t x}\, f(x)\, \mathrm{d}x.$$

(i) Show that $\widehat{f}$ is continuous.

(ii) Show that

$$\lim_{|t|\to\infty} \widehat{f}(t) = 0.$$

Problem 2.

(i) Determine the values of $a \in \mathbb{R}$ for which the function $g : \mathbb{R} \to \mathbb{R}$ given by

$$g(x) = \frac{1}{|x|^a\,(1+|x|)^{2a}}$$

belongs to $L^1(\mathbb{R})$.

(i) Does there exist a function f that belongs to $L^2([0,1])$ but does not belong to any other $L^p([0,1])$, for $1 \leq p \leq \infty$, $p \neq 2$? If yes, give an example. If not, justify.

(iii) Does there exist a function f that belongs to $L^2(\mathbb{R})$ but does not belong to any other $L^p(\mathbb{R})$, for $1 \leq p \leq \infty$, $p \neq 2$? If yes, give an example. If not, justify.

Problem 3. Let $d \geq 5$. Assume that for all functions $f \in C_c^{\infty}(\mathbb{R}^d)$ (infinitely differentiable functions with compact support in $\mathbb{R}^d$) the following inequality holds:

$$\|f\|_p \leq C\|\Delta f\|_2,$$

where C is a constant (independent of f) and Δ is the usual Laplacian $\sum_{i=1}^d \frac{\partial}{\partial x_i^2}$.

Find the value of p in terms of d.

(Hint: Do not try to prove the inequality; this you will learn later in your life. Fix a non-zero function f. Let $\delta > 0$ and consider the function $f_\delta(x) := f(\delta x)$ and analyze how both sides of the inequality depend on the parameter δ. Conclude from that.)

Problem 4. Let f and g be measurable non-negative functions on $\mathbb{R}^d$, and let m be the Lebesgue measure. For each $\alpha > 0$ let

$$E_\alpha = \{x \in \mathbb{R}^d \,:\, g(x) > \alpha\}.$$

Let $1 < p < \infty$ and assume that $f, g \in L^p(\mathbb{R}^d)$. Assume that for each $\alpha > 0$ we have

$$m(E_\alpha) \leq \frac{1}{\alpha} \int_{E_\alpha} f(y)\, \mathrm{d}y.$$

Show that

$$\|g\|_p \leq C\, \|f\|_p$$

for a constant C independent of f and g.

Problem 5. Let $f \in L^2(\mathbb{R}^2)$ and define

$$g_n(x) = (\phi_n * f)(x) = \int_{\mathbb{R}^2} \phi_n(x-y)\, f(y)\, \mathrm{d}y, \qquad \phi_n(x) = n^2\, x_1\, x_2\, e^{-n\pi|x|^2}, \quad (n \in \mathbb{N})$$

for all $x = (x_1, x_2) \in \mathbb{R}^2$, where $|x| = \sqrt{x_1^2 + x_2^2}$.

(i) Show that $\{g_n\}$ converges in $L^2(\mathbb{R}^2)$ as $n \to \infty$ and determine its limit.

(ii) If $\phi_n(x) = n^2\, |x_1|\, |x_2|\, e^{-n\pi|x|^2}$ instead, what would be such limit?

(iii) What can you say about the pointwise convergence of $\{g_n\}$ in the two situations above?

Problem 6. Let $\{h_n\}_{n\geq 1}$ be a sequence of non-negative and continuous functions in the interval $[0,1]$ and μ be a non-negative Borel measure on $[0,1]$. Let m be the Lebesgue measure and assume that

(i) $\lim_{n\to\infty} h_n(x) = 0$ a.e. with respect to m.

(ii) $\int_0^1 h_n\, \mathrm{d}x = 1$ for all n.

(iii) $\lim_{n\to\infty} \int_0^1 f\, h_n\, \mathrm{d}x = \int_0^1 f\, \mathrm{d}\mu$ for every continuous f on $[0,1]$.

Prove or disprove: $\mu \perp m$.