REAL ANALYSIS

ICTP - 2020

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- Problem 1 is worth an integer value in the interval $[-10, 10]$. Every other problem is worth a real value in the interval $[0, 5]$.
- Duration: 3h.

Problem 1. Mark (T) TRUE or (F) FALSE in each of the statements below. No need to justify your answer. Each correct answer is worth $+1$ point. Each wrong answer is worth -1 point. You may leave on or more items blank if you wish. Each item left blank is worth 0 point.

(   ) (a) If f and g are simple functions, then $h(x) = \max\{f(x), g(x)\}$ is also a simple function.

(   ) (b) Let $E \subset [0, 1]$ be an open set. Then χ_E is Riemann integrable.

(   ) (c) If $E \subset \mathbb{R}^d$ is Lebesgue measurable, there exists a sequence of compact sets $\{K_n\}_{n\geq 1}$ and a Lebesgue measurable set Z with $m(Z) = 0$ such that

$$E = \left(\bigcup_{n=1}^{\infty} K_n \right) \cup Z.$$

(   ) (d) If $\Omega = \{1, 2, 3, 4, 5\}$ there exists a $\sigma-$algebra Σ of Ω with exactly 13 subsets.

(   ) (e) Let (Ω, Σ, μ) be a measure space and f be a measurable function. If g is a function such that $g = f$ $\mu-$almost everywhere, then g is also measurable.

(   ) (f) Let $\Omega = \{1, 2, 3, 4\}$ and let Σ be the σ-algebra of all subsets of Ω. Let μ be a measure on (Ω, Σ) such that $\mu(\{1\}) = 1$, $\mu(\{1, 2\}) = 3$, $\mu(\{2, 3\}) = 6$ and $\mu(\{1, 3, 4\}) = 13$. Let $f : \Omega \to \mathbb{R}$ be given by $f(x) = x$. Then

$$\int_{\Omega} f \, \mathrm{d}\mu = 49.$$

(   ) (g) Let $\{E_k\}_{k\geq 1}$ be a sequence of Lebesgue measurable sets in $\mathbb{R}^d$ such that $\sum_{k=1}^{\infty} m(E_k) < \infty$. Then (Lebesgue)-almost every point in $\mathbb{R}^d$ belongs to at most finitely many E_k's.

(   ) (h) Let $A \subset \mathbb{R}^2$ be a Borel set. Then the set $A_0 = \{x \in \mathbb{R} : (x, 0) \in A\}$ is a Borel set of $\mathbb{R}$.

(   ) (i) Let $A \subset \mathbb{R}^2$ be a Lebesgue measurable set. Then the set $A_0 = \{x \in \mathbb{R} : (x, 0) \in A\}$ is a Lebesgue measurable set of $\mathbb{R}$.

(   ) (j) Consider the domain $\mathbb{R}$ with the $\sigma-$algebra of Lebesgue measurable sets. Let δ_x be the Dirac delta measure in $\mathbb{R}$ at the point x, i.e. $\delta_x(E) = 1$ if $x \in E$, and $\delta_x(E) = 0$ if $x \notin E$, for any measurable set E. Let $\{r_1, r_2, r_3, \ldots\}$ be an enumeration of the rationals and consider the measure

$$\nu = \sum_{n=1}^{\infty} \frac{1}{2^n} \delta_{r_n}.$$

Then ν is absolutely continuous with respect to the Lebesgue measure.

Problem 2. Let $\psi : \mathbb{R} \to \mathbb{R}$ be a bounded and continuous function such that

$$\left| \int_a^b \psi(x)\,\mathrm{d}x \right| \leq C \quad \text{for any} \quad a, b \in \mathbb{R}.$$

If $f \in L^1(\mathbb{R})$, consider the function $\widetilde{f}$ defined on $\mathbb{R}$ by

$$\widetilde{f}(y) := \int_{\mathbb{R}} f(x)\,\psi(yx)\,\mathrm{d}x$$

(i) Show that $\widetilde{f}$ is a continuous function.

(ii) Show that

$$\lim_{|y| \to \infty} \widetilde{f}(y) = 0.$$

Problem 3. Let δ be the Dirac delta measure in $\mathbb{R}$ at the point 0, i.e. $\delta(E) = 1$ if $0 \in E$, and $\delta(E) = 0$ if $0 \notin E$, for any Borel measurable set E. Consider the following two measures on the Borel subsets E of $\mathbb{R}$:

$$\mu_1(E) = \int_E e^{-\pi x^2}\,\mathrm{d}x;$$

$$\mu_2(E) = \delta(E) + m\big(E \cap [-1, 1]\big),$$

where m denotes the Lebesgue measure.

(i) Find the Lebesgue-Radon-Nikodym decomposition of μ_1 with respect to μ_2, and determine the Radon-Nikodym derivative of the absolutely continuous portion.

(ii) Do the same for μ_2 with respect to μ_1.

Problem 4. Let $\alpha > 1$. Let $f : \mathbb{R} \to \mathbb{R}$ be a Lebesgue measurable function such that

$$m\{x \in \mathbb{R} \,:\, |f(x)| > \lambda\} \leq \frac{1}{\lambda^\alpha} \quad \text{for all } \lambda > 0.$$

Let E be a Lebesgue measurable set with $m(E) < 1$. Find, with proper justification, the optimal inequality of the form

$$\int_E |f(x)|\,\mathrm{d}x \leq C\, m(E)^\beta, \tag{0.1}$$

where β may depend on α, and C is a constant (not depending on f or E). By optimal here we mean: the largest possible value of β and the smallest possible value of C.

(partial progress is given if you can show the integral in (0.1) is finite).

Problem 5. Let m be the Lebesgue measure and m^* be the exterior measure. For a generic set $X \subset \mathbb{R}^d$ let us define its *interior measure* as

$$m^i(X) := \sup_{\substack{F \subset X \\ F \text{ closed}}} m(F).$$

Let $E \subset \mathbb{R}^d$ be a Lebesgue measurable set with $m(E) < \infty$, and let $A \subset E$ be any set (not necessarily Lebesgue measurable).

(i) Show that $m(E) = m^i(A) + m^*(E - A)$.

(ii) Assuming that $m(E) = m^*(A) + m^*(E - A)$, show that A is Lebesgue measurable.