REAL ANALYSIS

ICTP - 2020

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Problem Set 1

Problem 1. If X is any collection of sets in Ω, then there exists a smallest $\sigma-$algebra Σ that contains X. In other words, show that the intersection of all $\sigma-$algebras that contain X is still a $\sigma-$algebra.

Problem 2. In order to verify that a function $f : \Omega \to \mathbb{R}$ is measurable show that it is sufficient to take intervals I of the form (a, ∞), for $a \in \mathbb{R}$ (note: there is nothing special attached to this form; one could also take intervals of the form $[a, \infty), (-\infty, a)$ or $(-\infty, a]$).

Problem 3. If f and g are measurable functions (with values on $\mathbb{R}$) and c is a real number, then cf , $f + g$, fg , $|f|$ are measurable.

Problem 4. If $\{f_n\}_{n\geq 1}$ is a sequence of measurable functions taking values on $\overline{\mathbb{R}}$ then

$$g_1(x) = \inf_n f_n(x) \; ; \; g_2(x) = \sup_n f_n(x) \; ; \; g_3(x) = \liminf_{n\to\infty} f_n(x) \; ; \; g_4(x) = \limsup_{n\to\infty} f_n(x)$$

are also measurable.

Problem 5. Verify that Problem 3 continues to hold for f and g taking values on $\overline{\mathbb{R}}$ if we set $0.(\pm\infty) = 0$ and $(\pm\infty) + (\mp\infty) = 0$.

Problem 6. Let (Ω, Σ, μ) be a measure space.

(i) If $A_1 \subset A_2 \subset \ldots \subset A_n \subset \ldots$ is an increasing sequence of measurable sets, then

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n).$$

(ii) If $A_1 \supset A_2 \supset \ldots \supset A_n \supset \ldots$ is a decreasing sequence of measurable sets, and $\mu(A_1) < +\infty$, then

$$\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n).$$

Note: The condition $\mu(A_1) < +\infty$ (or at least $\mu(A_n) < +\infty$ for some n) is important here; can you think of a counterexample if this condition is not assumed?

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Problem 7. Let (Ω, Σ, μ) be a measure space. Let $\mathcal{N} = \{N \in \Sigma\,;\, \mu(N) = 0\}$ and let

$$\overline{\Sigma} = \{A \cup B\,;\; A \in \Sigma \;\text{ and }\; B \subset N \;\text{ for some }\; N \in \mathcal{N}\}.$$

Show that $\overline{\Sigma}$ is a $\sigma-$algebra and there is a unique extension $\overline{\mu}$ of μ to a complete measure on $\overline{\Sigma}$.

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