REAL ANALYSIS

ICTP - 2020

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Problem Set 3

Problem 18. Find a sequence of real-valued non-negative functions $\{f_k\}_{k\geq 1}$ on $[0,1]$ so that

$$(a)\limsup_{k\to\infty} f_k(x) = \infty \ \ \forall x \in [0,1] \quad \text{and} \quad (b)\int_0^1 f_k(x)\,\mathrm{d}x \to 0.$$

Problem 19. Let (Ω, Σ) be a measurable space, and let $\{f_n\}_{n\geq 1}$ be a sequence of measurable functions. Suppose that for all $x \in \Omega$ the sequence $\{f_n(x)\}_{n\geq 1}$ is bounded. Let $E = \{x \in \Omega \ : \ (f_n(x))_{n\geq 1} \text{ converges}\}$. Show that $E \in \Sigma$.

Problem 20. For each $n \geq 1$ let $f_n : [0,1] \to \mathbb{R}$ be a continuous function. Suppose that

(i) $0 \leq f_n(x)$ for all n and all $x \in [0,1]$;
(ii) the sequence of partial sums $\sum_{n=1}^{N} f_n(x)$ converges uniformly on $[0,1]$ as $N \to \infty$;
(iii) the sum $\sum_{n=1}^{\infty}(\sup\{f_n(x) \ : \ 0 \leq x \leq 1\})$ diverges.

Show, by giving an example, that such a sequence of functions exists, or prove that such a sequence does not exist.

Problem 21. Let $F = (f_1, f_2, \ldots, f_d) : \mathbb{R}^d \to \mathbb{R}^d$ be a polynomial map, i.e. each $f_i \in \mathbb{R}[x_1, x_2, \ldots, x_d]$ for $i = 1, 2, \ldots, d$. Show that if $E \subset \mathbb{R}^d$ has Lebesgue measure zero then $F(E)$ also has Lebesgue measure zero.
(Hint: You may first try to prove that a Lipschitz map $G : \mathbb{R}^d \to \mathbb{R}^d$ takes sets of measure zero into sets of measure zero).

Problem 22. Construct a (non-measurable) function $f : \mathbb{R} \to \mathbb{R}$ with the following property: for any $g : \mathbb{R} \to \mathbb{R}$ such that

$$|g(x) - f(x)| < 1 \ \text{ for all } \ x \in \mathbb{R},$$

g is not measurable.

Problem 23. Let $f : [0,1] \to [0,1]$ be measurable, one-to-one and onto, satisfying the following condition:

$$\forall N \subset [0,1] \ , \ m(N) = 0 \implies m(f(N)) = 0.$$

Date: 30 de novembro de 2020.
2000 *Mathematics Subject Classification.* XX-XXX.
Key words and phrases. XXX-XXX.

Prove that $f^{-1} : [0,1] \to [0,1]$ is measurable.

Problem 24. We can construct generalized Cantor sets as follows. If I is a bounded interval and $\alpha \in (0,1)$, let us call the open interval with the same midpoint of I and length equal to α times the length of I the "open middle αth"of I. If $\{\alpha_j\}_{j\geq 1}$ is any sequence of numbers in $(0,1)$, we can define a decreasing sequence $\{K_j\}_{j\geq 0}$ of closed sets as follows. Let $K_0 = [0,1]$ and, for $j \geq 1$, K_j is obtained from K_{j-1} by removing the open middle α_jth of each of the intervals that make up K_{j-1}. The result limiting set

$$K = \bigcap_{j=0}^{\infty} K_j$$

is our generalized Cantor set (sometimes also called "fat"Cantor set). The usual Cantor set arises when taking $\alpha_j = \frac{1}{3}$ for all j. Show that

(i) K is compact, has empty interior, and is totally disconnected (i.e. the only connected subsets of K are single points). Moreover, K has no isolated points.
(ii) $m(K) = \prod_{j=1}^{\infty}(1-\alpha_j)$.
(iii) Show that $m(K) > 0$ if and only if $\sum_{j=1}^{\infty} \alpha_j < \infty$.
(iv) Given $\beta \in (0,1)$ exhibit a sequence $\{\alpha_j\}_{j\geq 1}$ such that $m(K) = \beta$.

Problem 25. A measurable function is called *essentially bounded* if there exists a finite number $\beta \geq 0$ such that the set $\{x \;:\; |f(x)| > \beta\}$ has measure zero. Prove or disprove the following proposition:
"Let $f : [0,1] \to \mathbb{R}$ be Lebesgue integrable. Then there exists $0 < a < b < 1$ such that f is essentially bounded in the interval (a,b)".

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