REAL ANALYSIS

ICTP - 2020

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Problem Set 4

Problem 26. Study the cases of equality in Hölder's and Minkowski's inequalities as follows.

(i) For $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, if $f \in L^p$ and $g \in L^q$ show that equality holds in
$$\|fg\|_r \leq \|f\|_p \, \|g\|_q$$
if and only if there exists $\alpha \geq 0$ such that $|g|^q = \alpha |f|^p \quad \mu -$ a.e. or if $f = 0$ $\mu-$a.e..

(ii) For $1 < p < \infty$ and $f, g \in L^p$, show that equality holds in
$$\|f + g\|_p \leq \|f\|_p + \|g\|_p$$
if and only if $g = \alpha f$ $\mu-$a.e. for some $\alpha \geq 0$ or $f = 0$ $\mu-$a.e.. What about if $p = 1$?

Problem 27. Let $I \subset \mathbb{R}$ be an open interval and $\varphi : I \to \mathbb{R}$ be a convex function, that is
$$\varphi\big((1-\lambda)x + \lambda y\big) \leq (1-\lambda)\varphi(x) + \lambda\varphi(y)$$
for all $x, y \in I$ and $0 \leq \lambda \leq 1$. Show that φ is continuous.

Problem 28. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be positive numbers with $\sum_{j=1}^n \alpha_j = 1$. Let $x_1, x_2, \ldots, x_n$ be non-negative real numbers. Show that
$$x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n} \leq \alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n.$$

Problem 29. Recall that a bounded function $f : I \to \mathbb{R}$ (with I interval) is Riemann integrable if and only if the set of points of discontinuity of f has measure zero (if you have never seen this, or have forgotten, go and read about it!).

Let $E \subset [0,1]$ be an open set, and let χ_E be its characteristic function.

Prove or disprove: χ_E is Riemann integrable.

Problem 30. If $f \in L^p(\mathbb{R}^d)$ and $v \in \mathbb{R}^d$, we define the translation $f_v(x) := f(x+v)$.

(i) If $1 \leq p < \infty$ show that translation is a continuous operator in $L^p(\mathbb{R}^d)$, that is, show that
$$\lim_{|v| \to 0} \|f_v - f\|_p = 0.$$

(ii) Show that this is not true when $p = \infty$.

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Problem 31. Let $g : \mathbb{R} \to \mathbb{C}$ be a bounded, measurable, periodic function with period 1, and let f be a function in $L^1(\mathbb{R})$. Show that

$$\lim_{n\to\infty} \int_{\mathbb{R}} f(x)\, g(nx)\, \mathrm{d}x = \left(\int_0^1 g(x)\, \mathrm{d}x\right)\left(\int_{\mathbb{R}} f(x)\, \mathrm{d}x\right).$$

Problem 32. Assume that $f \in L^\infty([0,1])$ (with respect to the Lebesgue measure). Show that

$$\lim_{p\to\infty} \|f\|_p = \|f\|_\infty.$$

Problem 33. Let $f \in L^1(\mathbb{R}^d)$. The for any $\varepsilon > 0$, show that there exists a $\delta > 0$ such that for every Lebesgue measurable set $E \subset \mathbb{R}^d$ with $m(E) < \delta$ we have

$$\left|\int_E f(x)\, \mathrm{d}x\right| < \varepsilon.$$

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