FUNCTIONAL ANALYSIS

ICTP - 2019

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Exam 1 - March 15, 2019.

- Choose 5 out of the following 6 problems.
- Each problem is worth 5 points.
- Duration: 2h.

Problem 1. If $f \in L^p(\mathbb{R}^d)$ and $g \in L^{p'}(\mathbb{R}^d)$, for some $1 \leq p \leq \infty$, show that $f * g$ is a continuous function.

Problem 2. Let E be a Banach space. Let $T : E \to E$ and $S : E^* \to E^*$ be linear operators such that $f(Tx) = (Sf)(x)$ for any $x \in E$ and $f \in E^*$. Show that T and S are continuous.

Problem 3. Let $f(x) = 1/(1 + |x|^2)$ and let f_n be defined as

$$f_n(x) = \int_{-\infty}^{\infty} n^{1/3}\, e^{-n|x-y|} f(y)\, \mathrm{d}y.$$

For each $\alpha \in \mathbb{R}$ determine $\lim_{n\to\infty} n^{\alpha} f_n$ in $L^2(\mathbb{R})$ (if it exists).

Problem 4. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function such that

$$\int_0^1 t^p\, f(t)\, \mathrm{d}t = 0$$

for all prime numbers p. Prove or disprove: $f \equiv 0$.

Problem 5. Let $\ell^2 = \{a = (a_1, a_2, a_3, ...); a_j \in \mathbb{R};\ \sum_{j=1}^{\infty} |a_j|^2 < \infty\}$ be the usual real vector space of square summable sequences, with norm given by

$$\|a\| = \left(\sum_{j=1}^{\infty} |a_j|^2 \right)^{1/2}.$$

Prove or disprove: For every $0 \leq C \leq 1$ there is a sequence $\{x_n\}$ of elements in ℓ^2 such that x_n converges weakly to some $x \in \ell^2$ and

$$\|x\| = C \liminf_{n\to\infty} \|x_n\|.$$

Problem 6. Let E be a reflexive Banach space and $K \subset E$ a closed and convex subset (non-empty). Let $x_0 \notin K$.

(a) Prove that there exists $y_0 \in K$ such that

$$\|x_0 - y_0\| = \inf_{y\in K} \|x_0 - y\|.$$

(b) Prove or disprove: y_0 is unique.

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