FUNCTIONAL ANALYSIS

ICTP - 2019

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Final Exam - April 24, 2019.

- Choose 5 out of the following 7 problems (you can solve them all if you want).
- Each problem is worth 5 points.
- Duration: 3h30m.

Problem 1. Let H be a Hilbert space. Let $\{u_n\}_{n\geq 1} \subset H$ and $u \in H$ be such that $u_n \rightharpoonup u$ weakly and $|u_n| \to |u|$ as $n \to \infty$. Prove or disprove: $u_n \to u$ strongly in H.

Problem 2. For $n \in \mathbb{N}$ let $f_n : \mathbb{R} \to \mathbb{C}$ be given by

$$f_n(x) = \frac{e^{2\pi i n x}}{1+x^2}.$$

Prove or disprove:

(i) f_n converges strongly in $L^2(\mathbb{R};\mathbb{C})$ as $n \to \infty$. If yes, what is the limit?

(ii) f_n converges weakly in $L^2(\mathbb{R};\mathbb{C})$ as $n \to \infty$. If yes, what is the limit?

Problem 3. Prove or disprove:

(i) $L^2(\mathbb{R})$ is isometrically isomorphic to $L^2([0,1])$.

(ii) $L^2(\mathbb{R})$ is isometrically isomorphic to $L^1(\mathbb{R})$.

(iii) For $1 < p < \infty$, $L^p(\mathbb{R})$ is isometrically isomorphic to $L^p([0,1])$.

Note: X is isometrically isomorphic to Y when there exists a bijective linear map $T : X \to Y$ that preserves the norm.

Problem 4. Let

$$K = \{u \in L^2([0,1])\,;\, |u(x)| \leq |x| \text{ a.e. in } [0,1]\}.$$

(i) Show that K is closed, convex and non-empty in $L^2([0,1])$.

(ii) Let $f(x) = -1 + 3x$, for $0 \leq x \leq 1$. Determine explicitly the projection $P_K f$. Justify.

Problem 5.

(i) Let E be a Banach space and $K \subset E$ be a convex set. Prove that K is closed in the strong topology if and only if K is closed in the weak topology.

(ii) State Mazur's lemma (on the weak convergence and convex combinations).

(iii) Let $1 < p < \infty$. Let $\{f_n\}_{n\geq 1} \subset L^p(\mathbb{R})$ be a bounded sequence. Assume that $f_n \to g$ pointwise almost everywhere, for some $g \in L^p(\mathbb{R})$. Prove or disprove: $f_n \rightharpoonup g$ weakly.

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Problem 6. Let X and Y be Banach spaces. Let $T : X \to Y$ be a bounded linear map and let $B_X = \{x \in X \ : \ \|x\|_X \leq 1\}$.

(i) Assume that $\overline{T(B_X)}$ is compact in Y (the closure here is with respect to the strong topology). Prove that if $x_n \rightharpoonup x$ weakly in X then $Tx_n \to Tx$ strongly in Y.

(ii) Assume that X is reflexive and that T satisfies the following property: if $x_n \rightharpoonup x$ weakly in X then $Tx_n \to Tx$ strongly in Y. Prove that $\overline{T(B_X)}$ is compact in Y.

Problem 7. Let

$$X = \big\{f \in L^2(\mathbb{R};\mathbb{C}) \cap C(\mathbb{R};\mathbb{C}) \ : \ \text{supp}(\widehat{f}) \subset [-\tfrac{1}{2}, \tfrac{1}{2}]\big\}.$$

In X we consider the usual $L^2(\mathbb{R};\mathbb{C})$-norm.

(i) Prove that X is a closed subspace of $L^2(\mathbb{R};\mathbb{C})$ (and hence it is a Hilbert space itself).

(ii) Let $w \in \mathbb{R}$ be fixed. Prove that the evaluation functional $f \mapsto f(w)$ is bounded in X.

(iii) By the Riesz representation theorem, there exists a function $K_w \in X$ such that $\langle f, K_w\rangle = f(w)$ for all $f \in X$. Determine K_w.

Note: Recall that the inner product here is given by

$$\langle f, g\rangle = \int_{\mathbb{R}} f(x)\, \overline{g(x)}\, \mathrm{d}x.$$

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