FUNCTIONAL ANLYSIS

ICTP - 2019

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Problem Set 5

Solve as many problems as you can from Brezis' book, chapters IV and V. In addition, take a look at the following problems:

Problem 41.

(i) Show that for each $\beta > 0$ we have:

$$e^{-\beta} = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} e^{-\frac{\beta^2}{4u}} \, \mathrm{d}u.$$

(ii) Deduce the Fourier transform of $f_\lambda(x) = e^{-2\pi\lambda|x|}$ in $\mathbb{R}^d$, where $\lambda > 0$.

Problem 42. Let f be a function in $C_c^2(\mathbb{R}^d)$. Prove that:

$$\sum_{\substack{i=1 \\ j=1}}^{d} \int_{\mathbb{R}^d} \left| \frac{\partial^2}{\partial x_i \partial x_j} f(x) \right|^2 \mathrm{d}x \le C \sum_{i=1}^{d} \int_{\mathbb{R}^d} \left| \frac{\partial^2}{\partial x_i \partial x_i} f(x) \right|^2 \mathrm{d}x.$$

What is the best constant C in this inequality?

Problem 43. Let α, β be multi-indices and a, b be real numbers. Suppose that for any $f \in \mathbb{C}_c^\infty(\mathbb{R}^d)$ we have

$$\| |\xi|^a \partial^\alpha \widehat{f} \|_{L^q(\mathbb{R}^d)} \le C \| |x|^b \partial^\beta f \|_{L^p(\mathbb{R}^d)}.$$

What is the relation between $a, b, \alpha, \beta, p, q, d$?

Note: Above we use $f(x)$ and $\widehat{f}(\xi)$.

Problem 44. The purpose of the following exercise is to show that $\mathcal{F}(L^1(\mathbb{R}^d)) \subsetneq C_0(\mathbb{R}^d)$.

(i) In dimension $d = 1$, let $f \in L^1(\mathbb{R})$ and suppose that $\widehat{f}$ is an odd function. Show that

$$\left| \int_1^a \frac{\widehat{f}(x)}{x} \, \mathrm{d}x \right| \le C,$$

for any $a > 1$, with C independent of a.

(ii) Find an odd function $g \in C_0(\mathbb{R})$ such that

$$\left| \int_1^a \frac{g(x)}{x} \, \mathrm{d}x \right|$$

is not bounded as $a \to \infty$.

Problem 45.

(i) Let $0 < \alpha < d$ and define $C_\alpha = \Gamma(\alpha/2)/\pi^{\alpha/2}$. Prove that the Fourier transform of $C_\alpha|x|^{-\alpha}$ is $C_{d-\alpha}|x|^{-d+\alpha}$ in the tempered distribution sense, that is, for every φ in the Schwartz class $\mathcal{S}$ we have

$$\int_{\mathbb{R}^d} C_{d-\alpha}|x|^{-d+\alpha}\widehat{\varphi}(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} C_\alpha|x|^{-\alpha}\varphi(x)\,\mathrm{d}x.$$

Hint: As we did for the Fourier transform of the exponential, it is possible to relate these functions with the Gaussian. Show that

$$\int_0^\infty e^{-\pi\lambda|x|^2}\,\lambda^{\beta-1}\,\mathrm{d}\lambda = (\pi|x|^2)^{-\beta}\,\Gamma(\beta),$$

for $\beta > 0$.

(ii) Compute explicitly $g(x) = |x|^{-\alpha} * |x|^{-\beta}$, where $0 < \alpha, \beta < d$, and $d < \alpha + \beta < 2d$.

Problem 46. Let f be a smooth and integrable function on $\mathbb{R}$ such that f and $\widehat{f}$ decay faster than any polynomial (that is, $x^k f(x) \to 0$ as $|x| \to \infty$ for any k, and $\xi^k \widehat{f}(\xi) \to 0$ as $|\xi| \to \infty$ for any k). Prove that f belongs to the Schwartz class.

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