FUNCTIONAL ANALYSIS

ICTP - 2020

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Problem Set 1

Problem 1. Let X and Y be normed vector spaces and $T : X \to Y$ be a linear map. Prove that the following are equivalent:

(i) T is continuous.
(ii) T is continuous at one point.
(iii) T is bounded.

Problem 2. Let E be a vector space over $\mathbb{R}$. Prove that E^* is a Banach space.

Problem 3. Use Zorn's lemma to prove the following:

(a) Every set X can be totally ordered.

(b) Every vector space E has a basis.

Note: Here a basis means an 'algebraic basis' (or Hamel basis), i.e. a set $\{e_i\}_{i\in I}$ such that each $x \in E$ can be written in a unique way as

$$x = \sum_{J\subset I} \lambda_i e_i, \quad \text{with} \quad J \subset I, \quad J \text{ finite.}$$

(c) Every subspace $X \subset E$ admits a complementary subspace $Y \subset E$ such that $E = X \oplus Y$, i.e. each $e \in E$ can be written in a unique way as $e = x + y$,with $x \in X$ and $y \in Y$.

Problem 4. Let E be a Banach space. Prove that the unit ball $B = \{x \in E\ ;\ \|x\| \leq 1\}$ is compact if and only if E has finite dimension.

Problem 5. Prove Hahn-Banach's Theorem in its complex version: let E be a vector space over $\mathbb{C}$ and $p : E \to \mathbb{R}$ a function such that:

(i) $p(ax) = |a|p(x)$ for all $a \in \mathbb{C}$ and $x \in E$;
(ii) $p(x + y) \leq p(x) + p(y)$ for any $x, y \in E$.

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Let $G \subset E$ be a subspace and g be a linear functional on G that verifies

$$|g(x)| \leq p(x)$$

for all $x \in G$. Then there exists a linear functional f on E that extends g and verifies

$$|f(x)| \leq p(x)$$

for all $x \in E$.

Problem 6. Brezis' book - Exercise 1.3.

Problem 7. Brezis' book - Exercise 1.4.

Problem 8. Let $\chi : [-1,1] \to \mathbb{R}$ be the characteristic function of the interval $[0,1]$, i.e.

$$\chi(x) = \begin{cases} 1, & \text{if } 0 \leq x \leq 1; \\ 0, & \text{if } -1 \leq x < 0. \end{cases}$$

Prove that there exists a continuous linear functional

$$\varphi : L^\infty([-1,1]) \to \mathbb{R}$$

with norm 2 such that $\varphi(\chi) = 1$ and $\varphi(f) = 0$ for any continuous $f : [-1,1] \to \mathbb{R}$.

Problem 9. Let E be a Banach space (over $\mathbb{R}$, say). Let $u, v \in E$ be such that

$$\|u\| = \|v\| = 1 \quad \text{and} \quad \|2u + v\| = \|u - 2v\| = 3.$$

Show that there exists a linear functional $\varphi \in E^*$ of norm 1 such that

$$\varphi(u) = \varphi(v) = 1.$$

Problem 10. Let $f, g : \mathbb{R} \to \mathbb{R}$ be two Lipschitz functions such that

- $f(x) \leq g(x)$ for all $x \in \mathbb{R}$.
- $f(x)$ and $g(x)$ both tend to zero as $|x| \to \infty$.
- $\text{Lip}(g) \leq \text{Lip}(f)$ (here $\text{Lip}(f)$ stands for the Lipschitz constant of f, i.e. $\text{Lip}(f) = \sup_{x,y} |f(x) - f(y)|/|x - y|$).
- In each connected component of the open set $\{x \in \mathbb{R}\ ;\ g(x) > f(x)\}$, the function g is convex.

Show that

$$\|g'\|_p \leq \|f'\|_p$$

for all $1 \leq p < \infty$.

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