FUNCTIONAL ANALYSIS

ICTP - 2020

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Final Exam - April 21, 2020.

- Each problem is worth 5 points.
- Duration: 3h30min.

Problem 1. Suppose $f \in L^1(\mathbb{R}^d)$ is such that $\widehat{f} = f$. Prove that $\|f\|_p \leq \|f\|_1$ for all $1 \leq p \leq \infty$.

Problem 2. Suppose the following inequality holds for all functions $f \in C_c^\infty(\mathbb{R}^d)$:

$$\|f\|_q \leq C\|\Delta f\|_p,$$

where C is a universal constant and $1 \leq p, q \leq \infty$. What should be the relation between p, q and d?

Problem 3. Prove or disprove (with justification) each of the statements below.

(i) ℓ^1 is reflexive.

(ii) Let H be a Hilbert space and $\{u_n\} \subset H$ be a sequence such that $u_n \rightharpoonup u$ and $\|u_n\| \to \|u\|$, for some $u \in H$. Then $u_n \to u$.

(iii) Let E be a Banach space and $\{u_n\} \subset E$ be a sequence such that $u_n \rightharpoonup u$ and $\|u_n\| \to \|u\|$, for some $u \in E$. Then $u_n \to u$.

Problem 4. Prove or disprove (with justification) each of the statements below.

(i) Let $1 \leq p < \infty$, $f \in L^p(\mathbb{R}^d)$ and $\{f_n\} \subset L^p(\mathbb{R}^d)$ be a sequence of functions such that $f_n \rightharpoonup f$ and $f_n \to g$ almost everywhere. Then $f = g$ almost everywhere.

(ii) Let $1 \leq p < \infty$, $f \in L^p(\mathbb{R}^d)$ and $\{f_n\} \subset L^p(\mathbb{R}^d)$ be a sequence of functions such that $f_n \to f$ almost everywhere and $\|f_n\|_p \to \|f\|_p$. Then $f_n \to f$ in the L^p-norm.

(iii) There exists $f \in C_c^\infty(\mathbb{R}^d)$ a non-negative function such that $\widehat{f}$ is also non-negative.

Problem 5.

(i) Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Then for $f \in L^p(\mathbb{R}^d)$, $g \in L^{p'}(\mathbb{R}^d)$ show that $f * g$ is a continuous function with

$$\lim_{|x| \to \infty} (f * g)(x) = 0.$$

(ii) Let A and B be measurable sets of $\mathbb{R}^d$ with positive Lebesgue measure. Prove that the set

$$A + B = \{a + b;\ a \in A \text{ and } b \in B\}$$

contains an open ball.

Problem 6. Let $Q = \{(x,y) \in \mathbb{R}^2;\ |x| < 1\ ;\ |y| < 1\}$ be the open unit cube in $\mathbb{R}^2$. For $\varepsilon > 0$ let

$$W_\varepsilon(x,y) = \frac{\partial^2}{\partial x^2}\sqrt{x^2+\varepsilon}.$$

If $f : Q \to \mathbb{R}$ is a continuous function, find the limit

$$\lim_{\varepsilon \to 0} \int_Q f(x,y) W_\varepsilon(x,y)\, \mathrm{d}x\, \mathrm{d}y.$$

Problem 7. For $g \in L^1_{loc}(\mathbb{R}^d)$ define the centered Hardy-Littlewood maximal function as

$$Mg(x) = \sup_{r>0} \frac{1}{m(B_r(x))} \int_{B_r(x)} |g(y)|\, \mathrm{d}y,$$

where $B_r(x) = \{y \in \mathbb{R}^d;\ |y-x| \leq r\}$ and $m(B_r(x))$ is the d-dimensional Lebesgue measure of this ball. Let $d \geq 3$ and consider $f_\alpha(x) = |x|^{-\alpha}$, where $0 < \alpha < d$. Show that

$$Mf_\alpha(x) = C_\alpha\, f(x)$$

for all $x \in \mathbb{R}^d$, where C_α is a constant.

Problem 8. Let ℓ^2 be the Hilbert space of square summable sequences $\{a_n\}_{n=1}^\infty$ with norm $\left(\sum_n |a_n|^2\right)^{1/2}$. Prove or disprove: "there are closed subspaces $\{H_t\,;\ 0 \leq t \leq 1\}$ of ℓ^2 such that $H_s \neq H_t$ if $s \neq t$, and $H_s \subset H_t$, if $s < t$, for any $0 \leq s, t \leq 1$."

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