REAL ANALYSIS

ICTP - 2020

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Problem Set 2

Problem 8. [Borel-Cantelli lemma] Let (Ω, Σ, μ) be a measure space. Let $\{E_k\}_{k=1}^{\infty}$ be a countable family of measurable sets of Ω such that

$$\sum_{k=1}^{\infty} \mu(E_k) < \infty.$$

Let $E = \{x \in \Omega \; : \; x \in E_k \text{ for infinitely many } k\}$. Show that:

(i) E is measurable.
(ii) $\mu(E) = 0$.

[Hint: $E = \bigcap_{n=1}^{\infty} \left(\bigcup_{k \geq n} E_k \right)$]

Problem 9. Let φ and ψ be simple functions in $\mathcal{M}^+$.

(i) If $c \geq 0$ then $\int c\,\varphi \,\mathrm{d}\mu = c \int \varphi \,\mathrm{d}\mu$.
(ii) $\int (\varphi + \psi) \,\mathrm{d}\mu = \int \varphi \,\mathrm{d}\mu + \int \psi \,\mathrm{d}\mu$.
(iii) If $\varphi \leq \psi$ then $\int \varphi \,\mathrm{d}\mu \leq \int \psi \,\mathrm{d}\mu$.
(iv) The map $A \mapsto \int_A \varphi \,\mathrm{d}\mu$ is a measure on Σ.

Problem 10. Show that:

(i) If $\{f_n\}$ is a finite or infinite sequence in $\mathcal{M}^+$ and $f = \sum_n f_n$ show that

$$\int f \,\mathrm{d}\mu = \sum_n \int f_n \,\mathrm{d}\mu.$$

(ii) If $f \in \mathcal{M}^+$, then $\int f \,\mathrm{d}\mu = 0$ if and only if $f = 0$ $\mu-$a.e.
(iii) If $\{f_n\}_{n \geq 1}$ and f are functions in $\mathcal{M}^+$ such that $f_n(x)$ converges monotonically increasing to $f(x)$ for $\mu-$a.e. $x \in \Omega$, show that the conclusion of the monotone convergence still holds, i.e. that

$$\int f \,\mathrm{d}\mu = \lim_{n \to \infty} \int f_n \,\mathrm{d}\mu.$$

(iv) If $\{f_n\}_{n \geq 1}$ and f are functions in $\mathcal{M}^+$ such that $f_n(x)$ converges to $f(x)$ for $\mu-$a.e. $x \in \Omega$, show that the conclusion of Fatou's lemma still holds, i.e. that

$$\int f \,\mathrm{d}\mu \leq \liminf_{n \to \infty} \int f_n \,\mathrm{d}\mu.$$

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(v) If $f \in \mathcal{M}^+$ show that the map $A \mapsto \int_A f \, \mathrm{d}\mu$ is a measure on Σ.

Problem 11. Show that:

(i) A function $f \in L^1$ if and only if $|f| \in L^1$. In this case
$$\left| \int f \, \mathrm{d}\mu \right| \leq \int |f| \, \mathrm{d}\mu.$$

(ii) If f is measurable and $g \in L^1$, and $|f| \leq |g|$, then $f \in L^1$.

(iii) If $f \in L^1$ and $c \in \mathbb{R}$ then $cf \in L^1$ and $\int cf \, \mathrm{d}\mu = c \int f \, \mathrm{d}\mu$.

(iv) If $f, g \in L^1$ then $f + g \in L^1$ and $\int (f+g) \, \mathrm{d}\mu = \int f \, \mathrm{d}\mu + \int g \, \mathrm{d}\mu$.

(v) If $f, g, \in L^1$ verify that the following three statements are equivalent:

- (a) $\int_A f \, \mathrm{d}\mu = \int_A g \, \mathrm{d}\mu$ for every $A \in \Sigma$;
- (b) $\int |f - g| \, \mathrm{d}\mu = 0$;
- (c) $f = g$ $\mu-$a.e.

Problem 12. Let $f : \mathbb{R} \to \mathbb{R}$ be a Lebesgue measurable function. Assume that there exists a positive constant c with the following property: If $A \subset \mathbb{R}$ is bounded and measurable, and f is bounded on A, then
$$\left| \int_A f(x) \, \mathrm{d}x \right| \leq c$$
Prove or disprove: f is Lebesgue integrable.

Problem 13. Let $A \subset [0, 1]$ be a closed set. For each positive integer N let $b(N)$ be the number of integers n such that $0 \leq n \leq N - 1$ and
$$\left[\frac{n}{N}, \frac{n+1}{N} \right] \cap A \neq \emptyset.$$
Show that
$$\lim_{N \to \infty} \frac{b(N)}{N} = m(A)$$
where m denotes the Lebesgue measure.

Problem 14. For $n = 1, 2, \ldots$ let $f_n : \mathbb{R} \to \mathbb{R}$ be a Lebesgue measurable function. Assume that the set $A_n = \{x \in \mathbb{R} \; : \; |f_n(x)| > n^{-2}\}$ has Lebesgue measure $m(A_n) \leq 2^{-n}$. Show that the series $\sum_{n=1}^{\infty} f_n(x)$ is absolutely convergent for almost every $x \in \mathbb{R}$.

Problem 15. Let E_k be a sequence of Lebesgue measurable sets in $\mathbb{R}^d$ such that $\sum_{k=1}^{\infty} m(E_k) < \infty$. Let F_j be the subset of $\mathbb{R}^d$ consisting of the points that belong to precisely j of the subsets $\{E_k\}_{k=1}^{\infty}$. Show that
$$\sum_{k=1}^{\infty} m(E_k) = \sum_{j=1}^{\infty} j \, m(F_j).$$

Problem 16. Prove that the collection $\mathcal{A}$ of finite disjoint unions of h-intervals is an algebra. Prove that the $\sigma-$algebra generated by $\mathcal{A}$ is $\mathcal{B}_{\mathbb{R}}$.

Problem 17. Let $F : \mathbb{R} \to \mathbb{R}$ be a non-decreasing and right-continuous function. If $(a_j, b_j]$ are disjoint h-intervals ($j = 1, 2, 3, \ldots, n$) let

$$\mu_0 \left(\bigcup_{j=1}^{n} (a_j, b_j] \right) = \sum_{j=1}^{n} \big(F(b_j) - F(a_j) \big)$$

and let $\mu_0(\emptyset) = 0$. Prove that μ_0 is well-defined and defines a pre-measure in $\mathcal{A}$.

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