# REAL ANALYSIS

# ICTP - 2020

INSTRUCTOR: EMANUEL CARNEIRO

## Problem Set 5: Due date Nov 05, 2020

**Problem 34.** Let $X$ be a normed vector space, say over $\mathbb{R}$. A *linear functional* is a function $L : X \to \mathbb{R}$ that satisfies $L(\lambda x + y) = \lambda L(x) + L(y)$ for any $x, y \in X$ and $\lambda \in \mathbb{R}$. A linear functional is said to be bounded when

$$\|L\|_{X^*} := \sup_{\substack{x \in X \\ \|x\|_X \leq 1}} |L(x)| < \infty.$$

(i) Prove that a linear functional $L$ is bounded if and only if it is continuous.

(ii) Let $X^*$ be the vector space of bounded linear functionals. Show that $\|\cdot\|_{X^*}$ defines a norm in this space.

(iii) Prove that $X^*$ with this norm is in fact a Banach space.

Remark: In general, to prove that $X^*$ is not only the $L \equiv 0$ functional, one uses the Hahn-Banach theorem (you can assume that $X^*$ is not trivial here to make the exercise interesting). In some case, like $X = L^p(\Omega, \Sigma, \mu)$ you already know that if one fixes $g \in L^{p'}$ then the map $f \mapsto \int_\Omega f g \, \mathrm{d}\mu$ belongs to $X^*$ by Hölder's inequality.

**Problem 35.** Let $\{f_n\} \subset L^1([0,1])$ with

(i) $f_n \to f$ a.e., $f \in L^1([0,1])$;

(ii) $\|f_n\|_1 \to \|f\|_1$.

Show that $f_n \to f$ in $L^1([0,1])$.

**Problem 36.** Let $B = \{f \in L^2(\mathbb{R}) \ : \ \|f\|_2 \leq 1\}$. Let $\{f_n\} \subset B$ be a sequence such that $f_n \to f$ a.e. in $\mathbb{R}$. Show that $f \in B$ and that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)\, g(x)\, \mathrm{d}x = \int_{-\infty}^{\infty} f(x)\, g(x)\, \mathrm{d}x$$

for all $g \in L^2(\mathbb{R})$.

**Problem 37.** In this problem we denote a point $x \in \mathbb{R}^2$ by $x = (x_1, x_2)$.

(i) Let $A \subset \mathbb{R}^2$ be a Borel set. Prove or disprove: the set

$$A_0 = \{x_1 \in \mathbb{R} \, : \, (x_1, 0) \in A\} \subset \mathbb{R}$$

is a Borel set of $\mathbb{R}$.

---

*Date*: 30 de novembro de 2020.
2000 *Mathematics Subject Classification.* XX-XXX.
*Key words and phrases.* XXX-XXX.

(ii) Define two measures $\mu_1$ and $\mu_2$ on the Borel subsets of $\mathbb{R}^2$ as follows. For a Borel set $A \subset \mathbb{R}^2$

$$\mu_1(A) = \big|A \cap \{x \in \mathbb{R}^2 \,:\, x_2 = 0\}\big|_{\mathcal{L}^1} + \int_A e^{-|x|^2}\,\mathrm{d}x,$$

and

$$\mu_2(A) = \big|A \cap \{x \in \mathbb{R}^2 \,:\, |x|^2 \leq 1\}\big|_{\mathcal{L}^2}.$$

Here $|\cdot|_{\mathcal{L}^1}$ is the one-dimensional Lebesgue measure and $|\cdot|_{\mathcal{L}^2}$ is the two-dimensional Lebesgue measure. Determine the Lebesgue-Radon-Nikodym decomposition of $\mu_1$ with respect to $\mu_2$.

**Problem 38.** Let $f : \mathbb{R}^d \to \mathbb{R}$ be a Lebesgue measurable function such that

$$m\{x \,:\, |f(x)| > \lambda\} \leq \frac{C}{\lambda^2}$$

for some fixed constant $C > 0$ and all $\lambda > 0$. Prove that there exists a constant $C_1$ such that for any Lebesgue measurable set $E \subset \mathbb{R}^d$ of finite and positive measure we have

$$\int_E |f(x)|\,\mathrm{d}x \leq C_1\sqrt{m(E)}$$

Note: Here $m$ denotes the Lebesgue measure.

**Problem 39.** Define three measures on the Borel subsets $E \subseteq [-2, 2]$ as follows:

(i) $\mu_1(E)$ is the Lebesgue measure.
(ii) $\mu_2(E)$ is the Lebesgue measure of $E \cap (0, 1)$.
(iii) $\mu_3(E) = \int_E x^2\,\mathrm{d}x$

For each pair $i \neq j$ determine the Lebesgue-Radon-Nikodym decomposition of $\mu_i$ with respect to $\mu_j$. That is, determine a function $f_{i,j}$ defined on $[-2, 2]$, and a measure $\nu_{i,j}$ defined on the Borel subsets of $[-2, 2]$ such that

$$\mu_i(E) = \int_E f_{i,j}(x)\,\mathrm{d}\mu_j(x) + \nu_{i,j}(E) \quad \text{for all Borel subsets } E \subset [-2, 2],$$

and such that the measures $\mu_j$ and $\nu_{i,j}$ are mutually singular.

ICTP - Strada Costiera 11, Trieste, Italy, 34151.
*Email address*: `carneiro@ictp.it`