**REAL ANALYSIS**

**ICTP - 2020**

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PROBLEM SET 6: DUE DATE NOV 12, 2020

**Problem 40.** Assume that $(\Omega, \Sigma, \mu)$ is a measure space and $f \in L^1(\Omega, \Sigma, \mu)$ satisfies $\|f\|_1 = 1$. Prove that

$$\int_E \log|f(x)|\mathrm{d}\mu(x) \leq -\mu(E)\log\mu(E)$$

for any measurable subset $E$ such that $0 < \mu(E) < \infty$.

**Problem 41.** Let $R$ be a (say, closed) rectangle in the plane. Assume that $R$ is the almost disjoint union of a finite number of smaller rectangles $\{R_j\}_{j=1}^N$, ($R$ and all $R_j$'s with sides parallel to the coordinate axes). Assume that each rectangle $R_j$ has at least one side with an integer length.

Prove or disprove: $R$ has one side with an integer length.

**Problem 42.** Let $L : \mathbb{R}^d \to \mathbb{R}^d$ be a linear transformation. If $E \subset \mathbb{R}^d$ is Lebesgue measurable show that:

(i) $L(E)$ is Lebesgue measurable.

(ii) we have
$$m(L(E)) = |\det L|\, m(E).$$

(in particular, the Lebesgue measure is invariant under rotations). For (ii) you can proceed by the following script, using Fubini's theorem:

(a) consider first the case $d = 2$ and $L$ a strictly upper triangular transformation of the following type $L(x, y) = (x + ay, y)$. Then
$$\chi_{L(E)}(x,y) = \chi_E(L^{-1}(x,y))) = \chi_E(x - ay, y).$$
Therefore
$$\begin{aligned} m(L(E)) &= \int_{\mathbb{R}}\left(\int_{\mathbb{R}} \chi_E(x-ay,y)\,\mathrm{d}x\right)\mathrm{d}y \\ &= \int_{\mathbb{R}}\left(\int_{\mathbb{R}} \chi_E(x,y)\,\mathrm{d}x\right)\mathrm{d}y \\ &= m(E), \end{aligned}$$
where we simply used the translation-invariance of the measure.

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(b) Similarly, $m(L(E)) = m(E)$ if $L$ is strictly lower triangular. In general, one can write $L = L_1 D L_2$ where $L_j$ are strictly (upper and lower) triangular and $D$ is diagonal. Conclude.

(c) Extend the argument above to general dimension $d$.

**Problem 43.** We say that a sequence $\{f_k\}$ of measurable functions in $\mathbb{R}^d$ is *Cauchy in measure* if for every $\varepsilon$ we have

$$m\big(\{x \in \mathbb{R}^d \,:\, |f_k(x) - f_\ell(x)| > \varepsilon\}\big) \to 0 \quad \text{as } k, \ell \to \infty.$$

We say that $f_k$ *converges in measure* to a measurable function $f$ if for every $\varepsilon > 0$ we have

$$m\big(\{x \in \mathbb{R}^d \,:\, |f_k(x) - f(x)| > \varepsilon\}\big) \to 0 \quad \text{as } k \to \infty.$$

Prove the following:

(i) If $\{f_k\}$ is Cauchy in measure, then there is a measurable function $f$ to which this sequence converges in measure.

(ii) If $f_k$ converges to $f$ in measure, show that there exists a subsequence $f_{n_k}$ that converges to $f$ pointwise almost everywhere. Passing to a subsequence is really necessary here?

(iii) If $f_k \to f$ in $L^p$, then $f_k$ converges to $f$ in measure. Is the converse true? (say, assuming that all $f_k$ and $f$ are in $L^p$).

**Problem 44.** Let $X$ the vector space of (finite) signed measures over $(\Omega, \Sigma)$. Show that

(i) The function $\|\cdot\| : X \to [0, \infty)$ given by

$$\|\nu\| = |\nu|(\Omega)$$

defines a norm on $X$ (recall that $|\nu|$ is the total variation measure associated to $\nu$).

(ii) Show that $(X, \|\cdot\|)$ is a Banach space.

**Problem 45.** Let $f : \mathbb{R}^d \to \mathbb{R}$ be a measurable function in $L^1_{\text{weak}}(\mathbb{R}^d)$, i.e.

$$\theta = \sup_{\lambda > 0} \lambda \,.\, m\big(\{x \in \mathbb{R}^d \,:\, |f(x) > \lambda\}\big) < \infty.$$

(here $m$ denotes the Lebesgue measure). Prove that for every $\delta \in (0,1)$, and for every measurable set $E \subset \mathbb{R}^d$, $m(E) < \infty$, we have

$$\int_E |f(x)|^\delta \, \mathrm{d}x \le C\, m(E)^{1-\delta}\, \theta^\delta,$$

with an appropriate constant $C$ that is independent of $f$ and $E$.

**Problem 46.** Let $\{f_n\}_{n=1}^\infty \subset L^2([0,1])$. Assume that $\sup_n \|f_n\|_2 < \infty$ and let $1 \le p < 2$. Let $f : [0,1] \to \mathbb{R}$ and assume that $f_n \to f$ pointwise a.e. Prove that $\lim_{n\to\infty} \|f_n - f\|_p = 0$. Give an example to show that this is false for $p = 2$.

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