REAL ANALYSIS

ICTP - 2020

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Problem Set 7: Due date Dec 03, 2020

Problem 47. (Minkowski's inequality for integrals, aka. triangle inequality) Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be $\sigma-$finite measure spaces, and let f be a measurable function in the product space $\Omega_1 \times \Omega_2$. For $1 \leq p \leq \infty$ show that

$$\left(\int_{\Omega_1}\left(\int_{\Omega_2}|f(x,y)|\,\mathrm{d}\mu_2(y)\right)^p \mathrm{d}\mu_1(x)\right)^{1/p} \leq \int_{\Omega_2}\left(\int_{\Omega_1}|f(x,y)|^p\,\mathrm{d}\mu_1(x)\right)^{1/p}\mathrm{d}\mu_2(y).$$

Problem 48. Let $f : [0,1] \to \mathbb{R}$ be an absolutely continuous function such that

$$\int_0^1 f(x)\,\mathrm{d}x = 0.$$

Show that

$$\sup_{0\leq y\leq 1}\left|\int_0^1 (x-y)\,f'(x)\,\mathrm{d}x\right| \leq \sup_{0\leq x\leq 1}|f(x)|.$$

Problem 49. Let $f : \mathbb{R} \to \mathbb{R}$ be an absolutely continuous function with $f' \in L^1(\mathbb{R})$. Show that for all $h \in \mathbb{R}$ we have:

$$\int_{-\infty}^{\infty}|f(x+h) - f(x)|\,\mathrm{d}x \leq C\,|h|,$$

where C is a constant independent of h.

Problem 50. Let $f : [0,1] \to \mathbb{R}$ be an absolutely continuous function such that $f' \in L^2([0,1])$. Show that for each $\varepsilon > 0$ there exists $\delta > 0$ such that: if $0 \leq y < x \leq 1$ with $|x-y| < \delta$ then

$$\frac{|f(x)-f(y)|^2}{|x-y|} < \varepsilon.$$

Problem 51. Let φ a Lebesgue measurable function in $\mathbb{R}^d$ that verifies the following property: for each $d-$dimensional rectangle Q we have

$$\left|\int_Q \varphi(x)\,\mathrm{d}x\right| \leq \frac{M\,m(Q)}{1+m(Q)},$$

Date: 30 de novembro de 2020.
2000 *Mathematics Subject Classification.* XX-XXX.
Key words and phrases. XXX-XXX.

where M is a constant. Show that for each $f \in L^1(\mathbb{R}^d)$ we have

$$\lim_{k\to\infty} \int_{\mathbb{R}^d} \varphi(kx)\, f(x)\, \mathrm{d}x = 0.$$

Problem 52. Let $f_n : \mathbb{R} \to [0,\infty)$ be non-decreasing functions for each $n \in \mathbb{N}$. Suppose that for all $x \in \mathbb{R}$ we have

$$f(x) = \sum_{n=1}^{\infty} f_n(x) < \infty.$$

Prove that

$$f'(x) = \sum_{n=1}^{\infty} f_n'(x) \;\; \text{a.e.}$$

Problem 53. Given a function $g \in L^1_{loc}(\mathbb{R}^d)$ we define its centered Hardy-Littlewood maximal operator $\mathcal{M}g$ as

$$\mathcal{M}g(x) = \sup_{r>0} \frac{1}{m(B_r(x))} \int_{B_r(x)} |g(y)|\, \mathrm{d}y,$$

where $B_r(x) = \{y;\ |y-x| \leq r\}$ and $m(B_r(x))$ is the $d-$dimensional Lebesgue measure of this ball. Suppose that $d \geq 3$ and consider the function

$$f_\alpha(x) = |x|^{-\alpha},$$

where $0 < \alpha < d$.

(a) Show that
$$\mathcal{M}f_\alpha(x) = C_{\alpha,d}\, f(x)$$
for all $x \in \mathbb{R}^d$, where $C_{\alpha,d}$ is a constant (that may depend on α and d).

(b) (Extra credit) For each α, with $0 < \alpha < d$, decide if $C_{\alpha,d} = 1$ or $C_{\alpha,d} > 1$. Justify.

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