REAL ANALYSIS

ICTP - 2020

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PROBLEM SET 8: DUE DATE DEC 03, 2020

Problem 54. Let $f(x) = |x|^{-2}(1+|x|^2)^{-1}$ for $x \in \mathbb{R}^3$, with the usual Lebesgue measure.

(i) For which values of $p \in (0,\infty]$, do we have $f \in L^p(\mathbb{R}^3)$?

(ii) Let $g \in L^2(\mathbb{R}^3)$. Prove that $f * g \in L^2(\mathbb{R}^3)$ and determine an explicit constant C such that

$$\|f * g\|_2 \leq C\|g\|_2.$$

Problem 55. Let $f \in L^2(\mathbb{R}^2)$ and define

$$g_n(x) = (\phi_n * f)(x) = \int_{\mathbb{R}^2} \phi_n(x-y)\, f(y)\, \mathrm{d}y, \qquad \phi_n(x) = n^4\, x_1\, x_2\, e^{-n|x|},$$

for all $x = (x_1, x_2) \in \mathbb{R}^2$, where $|x| = \sqrt{x_1^2 + x_2^2}$.

(i) Show that $\{g_n\}$ is a Cauchy sequence in $L^2(\mathbb{R}^2)$ and determine its limit.

(ii) If $\phi_n(x) = n^4\, |x_1|\, |x_2|\, e^{-n|x|}$ instead, what would be such limit?

Problem 56. The Gauss-Weierstrass kernel $K : \mathbb{R}^d \to \mathbb{R}$ is defined as

$$K(x) = \frac{1}{(4\pi)^{d/2}} e^{-|x|^2/4}.$$

For $t > 0$ define $K_t(x) = t^{-d/2} K(x/\sqrt{t})$. Let $f \in L^p(\mathbb{R}^d)$, with $1 \leq p \leq \infty$. Prove that $u(x,t) = f * K_t(x) \in C^\infty(\mathbb{R}^d \times (0,\infty))$ and verifies

$$u_t - \Delta u = 0 \text{ in } \mathbb{R}^d \times (0,\infty),$$

with

$$\lim_{t\to 0^+} u(x,t) = f(x) \text{ a.e. } x \in \mathbb{R}^d.$$

Problem 57. The Poisson kernel $P : \mathbb{R}^d \to \mathbb{R}$ is defined as

$$P(x) = \frac{C_d}{(1+|x|^2)^{(d+1)/2}},$$

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where $C_d = \Gamma\left(\frac{d+1}{2}\right)/\pi^{(d+1)/2}$. For $y > 0$ define $P_y(x) = y^{-d}P(x/y)$. Let $f \in L^p(\mathbb{R}^d)$, with $1 \leq p \leq \infty$. Prove that $u(x,y) = f * P_y(x)$ is harmonic in $\mathbb{R}^{d+1}_+ = \{(x,y);\ x \in \mathbb{R}^d;\ y > 0\}$ and that

$$\lim_{y \to 0^+} u(x,y) = f(x) \ \text{ a.e. } x \in \mathbb{R}^d.$$

Problem 58. For $y > 0$ define

$$\varphi_y(x) = \frac{1}{\pi}\sin\left(\frac{\pi x y}{x^2+y^2}\right)\frac{y}{x^2+y^2};$$

$$\varphi_y * f(x) = \int_{-\infty}^{\infty} \varphi_y(x-z)\, f(z)\, \mathrm{d}z, \quad f \in L^2(\mathbb{R}).$$

(i) Show that $K(x,y) = \sqrt{y}\,(\varphi_y * f)(x)$ is a bounded function of x and y with $(x,y) \in \mathbb{R} \times (0,\infty)$.

(ii) For each $y > 0$, show that $\varphi_y * f \in L^2(\mathbb{R})$. Show also that $\varphi_y * f$ converges in $L^2(\mathbb{R})$ as $y \to 0$ and find this limit.

Problem 59. Let A and B be measurable subsets of $\mathbb{R}$, such that $m(A) > 0$ and $m(B) > 0$. Show that the set

$$A + B = \{x + y;\, x \in A,\, y \in B\}$$

contains an interval.

Problem 60. Let $I = [-M, M] \subset \mathbb{R}$ and let $h : \mathbb{R} \to \mathbb{R}$ be a continuous function with support in I. Define, for $z \in \mathbb{C}$,

$$g(z) = \int_I \frac{1}{t-z}\, h(t)\, \mathrm{d}t$$

(i) Prove that g is analytic in $\mathbb{C} \setminus I$.

(ii) Find the limit $\lim_{|z| \to \infty} z g(z)$.

(iii) For $\varepsilon > 0$ and $\sigma \in \mathbb{R}$, find the limit

$$\lim_{\varepsilon \to 0}\{g(\sigma + i\varepsilon) - g(\sigma - i\varepsilon)\}.$$

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